A parallel between extended formal concept analysis and bipartite graphs analysis

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Presentation outline

1. Reading FCA as a Bipartite Graph Analysis
   - Formal Context ⇔ Bipartite Graph.
   - Formal Concept ⇔ Maximal Bi-Clique
   - (smallest) Conceptual World ⇔ Connected Components
2. More flexible clustering ?
   - Be more flexible, why ?
   - Use short random walk to fuzzify the context ?
   - Parallel with random walks methods
3. Conclusion

Data model: **Formal Context**

- **O**: a set of objects (here: 1, 2, 3, 4, 5, 6, 7, 8),
- **P**: a set of properties (here: a, b, c, d, e, f, g, h, i),
- R: a *binary relation* between **O** and **P**.

Objects

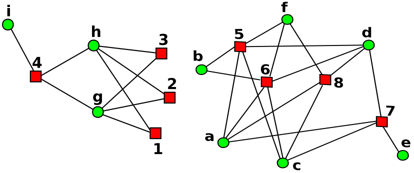
	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

a **Formal Context** is a **Bipartite Graph**.

Objects

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

⇔



$$\mathcal{R} = (\mathbf{O}, \mathbf{P}, R)$$
$$\mathcal{G} = (V_o, V_p, E)$$

- $\mathbf{O} \Leftrightarrow V_o$, set of "o-vertices"
- $\mathbf{P} \Leftrightarrow V_p$, set of "p-vertices"
- $R \Leftrightarrow E$, set of edges

$$E \subseteq V_o \times V_p$$

Intent & Extent operators

Objects

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

Intent operator: $(.)^{\Delta}$

$X \in \mathbf{O}$ (ex: $X = \{3,4\}$)

$X^{\Delta} = \{y \in \mathbf{P} \mid \forall x \in X, (x,y) \in R\}$

$\emptyset^{\Delta} = \mathbf{P}$ (ex: $\{3,4\}^{\Delta} = \{g,h\}$)

Extent operator: $(.)^{-1\Delta}$

$Y \in \mathbf{P}$ (ex: $Y = \{c,d,f\}$)

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$\emptyset^{-1\Delta} = \mathbf{O}$ (ex: $\{c,d,f\}^{-1\Delta} = \{5,6,8\}$)

- X^{Δ} is the set of properties possessed by all objects in X.
- $Y^{-1\Delta}$ is the set of objects having all properties in Y.

Intent & Extent operators

Objects

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
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g	×	×	×	×				
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Graph reading of Intent & Extent

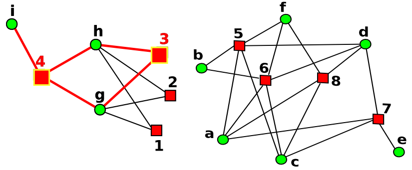
$\Gamma(x)$ = neighbours of x

$$X^{\Delta} = \bigcap_{x \in X} R(x) = \bigcap_{x \in X} \Gamma(x)$$

- X^{Δ} is the intersection of neighbours of vertices of X.

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

⇔



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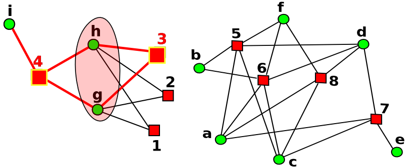
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

⇔



Formal Concept !

Formal Concept

A pair (X, Y) such that : $Y = X^{\Delta}$ et $X = Y^{-1\Delta}$

- X is its **extent**, ie. the set of objects having all properties in Y
- Y is its **intent**, ie. the set of properties shared by all objects in X.

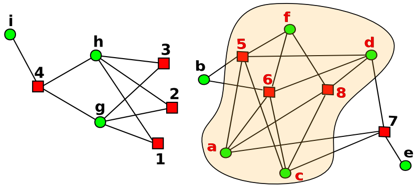
	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

Formal Concept ⇔ Maximal Bi-Clique

Proposition

$X = Y^{-1\Delta}$ and $Y = X^{\Delta}$, iff $X \cup Y$ is a maximal bi-clique.

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				



Possibility th. Operators

(Extended) Formal Concept Analysis

- (potential) possibility

$$X^{\Pi} = \{y \in \mathbf{P} | R^{-1}(y) \cap X \neq \emptyset\} \quad (1)$$

Properties possessed by at least one object in X.

- (actual) necessity

$$X^{N} = \{y \in \mathbf{P} | R^{-1}(y) \subseteq X\} \quad (2)$$

Properties possessed only by objects in X.

- *guaranteed possibility* [classical intent/extent]

$$X^{\Delta} = \{y \in \mathbf{P} | R^{-1}(y) \supseteq X\} \quad (3)$$

Properties possessed by all objects in X.

- potential necessity

$$X^{\nabla} = \{y \in \mathbf{P} | R^{-1}(y) \cup X \neq \mathbf{O}\} \quad (4)$$

Properties not possessed by all objects outside of X.

[Dubois et al., 2007, Dubois and Prade, 2009]

New Galois connections ?

(Extended) Formal Concept Analysis

These new operators lead to consider the following Galois connections:

- the pairs (X, Y) such that $X^{\Pi} = Y$ and $Y^{-1\Pi} = X$;
- the pairs (X, Y) such that $X^{N} = Y$ and $Y^{-1N} = X$;
- the pairs (X, Y) such that $X^{\nabla} = Y$ and $Y^{-1\nabla} = X$.

but : $$(X^{\nabla} = Y \text{ and } Y^{-1\nabla} = X) \quad \Leftrightarrow \quad (X^{\Delta} = Y \text{ and } Y^{-\Delta} = X)$$

it's **formal concept** !

and : $$(X^{\Pi} = Y \text{ and } Y^{-1\Pi} = X) \quad \Leftrightarrow \quad (X^{N} = Y \text{ and } Y^{-1N} = X)$$

it's something else ...

A new Galois connection: "conceptual world"

(Extended) Formal Concept Analysis

Let be a pair (X, Y) such that $X^{\sqcap} = Y$ and $Y^{-1\sqcap} = X$.

Independent sub-contexts or **Conceptual Worlds**

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

ex: $X = \{1, 2, 3, 4\}$ and $Y = \{g, h, i\}$

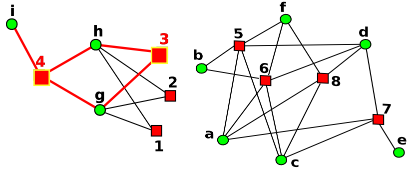
Graph reading of $(.)^{\sqcap}$

$$X^{\sqcap} = \bigcup_{x \in X} R(x) = \bigcup_{x \in X} \Gamma(x)$$

- $X^{\sqcap}$ is the union of neighbours of vertices of X.

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

⇔



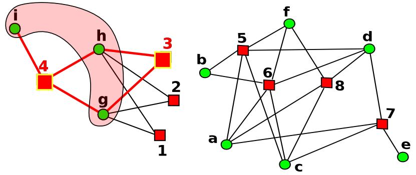
Graph reading of $(.)^{\Pi}$

$$X^{\Pi} = \bigcup_{x \in X} R(x) = \bigcup_{x \in X} \Gamma(x)$$

- X^{Π} is the union of neighbours of vertices of X.

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				

⇔



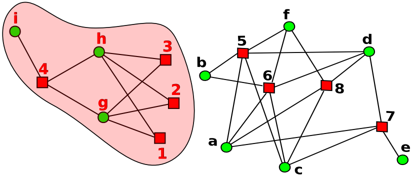
(smallest) Conceptual World ⇔ Connected Components

Proposition

For a pair (X, Y) the two following propositions are equivalent:

1. $X = Y^{-1\sqcap}$ *and* $Y = X^{\sqcap}$ *and* *there is no* $X' \subset X$ *and* $Y' \subset Y$ *s.t.* $X' = Y'^{-1\sqcap}, Y' = X'^{\sqcap}$.
2. $X \cup Y$ *is a maximal connected component* *(and counts at least 2 vertices).*

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c					×	×	×	×
d					×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h	×	×	×	×				
i				×				



- Formal Context ⇔ Bipartite Graph

- Formal Concepts ⇔ Maximal Bi-Cliques
- Conceptual Worlds (smallest ones) ⇔ Connected Components

Extreme definitions of what a graph cluster could be:

1. a group of vertices with **no link missing inside**.
2. a group of vertices with **no link with outside**.

definitions already pointed out in graph literature : [Schaeffer, 2007]

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	1	2	3	4	5	6	7	8
a					×	×	×	×
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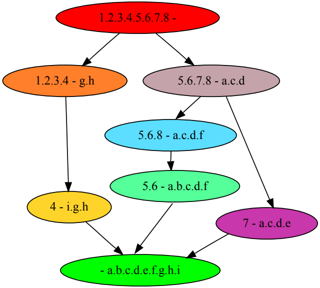

	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
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h	×	×	×	×				
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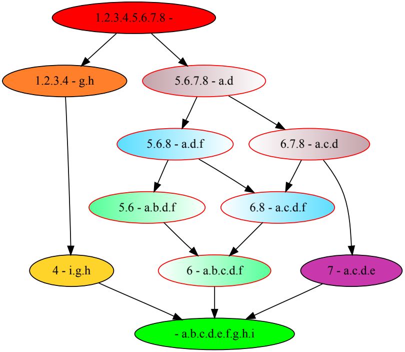

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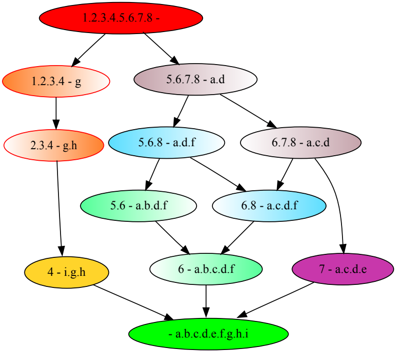
h		×	×	×				
i				×				



	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c						×	×	×
d				×	×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h		×	×	×				
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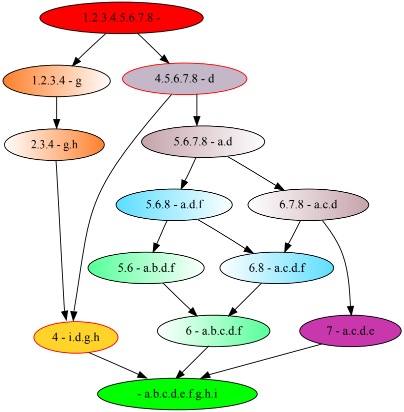

- How to deal with a **noisy** binary relation ?
  - link missing ⇒ could splits some concepts in two,
  - false link ⇒ could merges two conceptual worlds in one.
- How to **summarize** a concept lattice ?
  - Forget some (unimportant) edges to create two conceptual worlds,
  - Add missing edges to create a more general concept.

Same issue ? we can consider as "noise" some links in order to summarize...

R is still a binary relation !

Can we use some graph clustering method ?

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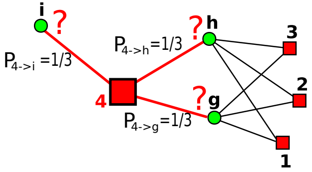
Can we use some graph clustering method ?

Using random walks to define graph clusters

Random walkers tend to be trapped into clusters...



- [Gaume, 2004], distance between vertices using random walks:
  - Compute a short length random walk from each vertex: $C(v,t)$
  - $d(v1,v2) = d(C(v1,t), C(v2,t))$
  - Distance used to detect clusters: [Pons and Latapy, 2006]
- [Rosvall and Bergstrom, 2008],[Delvenne et al., 2008] :
  Quality measure : a good partition (of graph vertices) is a partition such that a random walker **remains** into clusters.

Using random walks to define graph clusters

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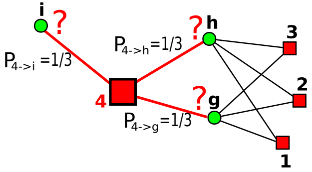

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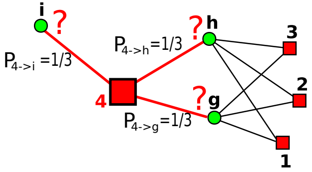

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R	1	2	3	4	5	6	7	8
a					×	×	×	×
b					×	×		
c						×	×	×
d				×	×	×	×	×
e							×	
f					×	×		×
g	×	×	×	×				
h		×	×	×				
i				×				

$$R^P_{(x,y)} \leftarrow \frac{1}{2}.(P^3_{x\to y} + P^3_{y\to x})$$

$P^3_{x\to y}$: probability to go from x to y in 3 steps.

α-cut: $R_{(x,y)} \geq 0.16$

R^P	1	2	3	4	5	6	7	8
a	0	0	0	0.05	0.22	0.25	0.23	0.22
b	0	0	0	0.03	0.22	0.23	0.10	0.16
c	0	0	0	0.04	0.16	0.21	0.23	0.20
d	0.04	0.05	0.05	0.14	0.21	0.23	0.22	0.21
e	0	0	0	0.03	0.07	0.12	0.28	0.12
f	0	0	0	0.04	0.23	0.25	0.14	0.20
g	0.35	0.37	0.37	0.32	0.01	0.01	0.01	0.01
h	0.21	0.30	0.30	0.30	0.01	0.01	0.01	0.01
i	0.06	0.11	0.11	0.28	0.03	0.03	0.03	0.03

$$R^P_{(x,y)} \leftarrow \frac{1}{2}.(P^3_{x\rightarrow y} + P^3_{y\rightarrow x})$$

$P^3_{x\rightarrow y}$: probability to go from x to y in 3 steps.

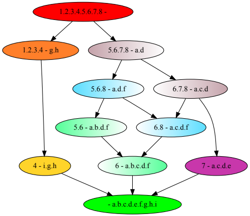
α-cut: $R_{(x,y)} \geq 0.16$

R^P	1	2	3	4	5	6	7	8
a	0	0	0	0.05	0.22	0.25	0.23	0.22
b	0	0	0	0.03	0.22	0.23	0.10	0.16
c	0	0	0	0.04	0.16	0.21	0.23	0.20
d	0.04	0.05	0.05	0.14	0.21	0.23	0.22	0.21
e	0	0	0	0.03	0.07	0.12	0.28	0.12
f	0	0	0	0.04	0.23	0.25	0.14	0.20
g	0.35	0.37	0.37	0.32	0.01	0.01	0.01	0.01
h	0.21	0.30	0.30	0.30	0.01	0.01	0.01	0.01
i	0.06	0.11	0.11	0.28	0.03	0.03	0.03	0.03

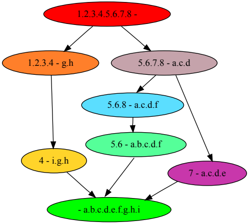
$$R^P_{(x,y)} \leftarrow \frac{1}{2}.(P^3_{x\to y} + P^3_{y\to x})$$

$P^3_{x\to y}$: probability to go from x to y in 3 steps.

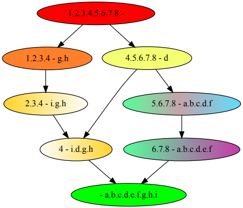
α-cut: $R_{(x,y)} \geq 0.16$



R^P_α, $\alpha = 0.20$



R^P_α, $\alpha = 0.16$



R^P_α, $\alpha = 0.10$

- It is a good example... in which conditions it works ?
- What is the good cut ? apply fuzzy FCA ?
- implementation... complexity...

Formal parallel with random walk methods

$$X^P(y) = \sum_{x \in \mathbf{O}} X(x).P_{x \to y}$$

with :

- $X^P(y)$: probability to be in y at time $t+1$,
- $X(x)$: probability to be in x at a time t.

$$X^\Delta(y) = \min_{x \in \mathbf{O}} X(x) \to R(x,y)$$

$$X^\Pi(y) = \max_{x \in \mathbf{O}} X(x) * R(x,y)$$

Interesting sets are stable by these operators...

Future works

This is clearly a preliminary work !

- Use random walks to fuzzify a binary relation ?
- Re-define intent/extent operators in non-crisp manner ?
  ("almost" instead of "all")
- Use knowledge about "real world network" (or *complex networks*) properties to develop heuristics for FCA ?
  (small world effect, power law degree distribution, etc...)

Thank you !

Questions ?

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